MATHEMATICS OF COMPUTATION
VOLUME 45, NUMBER 172
OCTOBER 1985, PAGES 621-632

A Counterexample to a Conjecture of Mahler on Best P-Adic Diophantine Approximation Constants

By Alice A. Deanin

Abstract. In 1940, Mahler proposed a conjecture regarding the value of best P-adic Diophantine approximation constants. In this paper, a computational technique which tests the conjecture for any particular P is described. A computer search verified the conjecture for all $P \leqslant 101$, except 83. The case $P = 83$ is discussed. A counterexample is given.

1. Introduction to Mahler's Algorithm and Mahler's Conjecture. Mahler [5] presented an algorithm which yields sequences of approximations to P-adic integers by rational numbers which are best with respect to a real reduced positive-definite binary quadratic form of determinant -1. Let λ be a fixed complex number in F, the fundamental domain of the modular group, i.e., the set of complex numbers in the upper half-plane which satisfy

$$-\tfrac{1}{2} \leqslant \operatorname{Re} z < \tfrac{1}{2} \text{ and } |z| > 1 \quad \text{or} \quad -\tfrac{1}{2} \leqslant \operatorname{Re} z \leqslant 0 \text{ and } |z| = 1.$$

Set

$$\Phi(X, Y) = \frac{1}{\operatorname{Im} \lambda}(X - \lambda Y)(X - \bar{\lambda} Y) = \frac{1}{\operatorname{Im} \lambda}|X - \lambda Y|^2.$$

Let ζ be a P-adic integer. For every $n > 0$, let A_n be the unique (rational) integer which satisfies

$$0 \leqslant A_n < P^n \quad \text{and} \quad \zeta \equiv A_n \pmod{P^n}.$$

The algorithm defines three sequences. The first of these, $z(\zeta)$, is a sequence of complex numbers. Let $z_n \in z(\zeta)$ be defined as the unique complex number in F which is equivalent by an element of the modular group to $(A_n + \lambda)/P^n$. That is, for some integer matrix

$$\begin{bmatrix} r_n & r_n' \\ q_n & q_n' \end{bmatrix}$$

of determinant 1, $(A_n + \lambda)/P^n = (r_n z_n + r_n')/(q_n z_n + q_n')$.

The second sequence $T(\zeta)$ is a sequence of 2×2 integer matrices where, for each $n \geqslant 0$, $T_n \in T(\zeta)$ is defined by

$$T_n = \begin{bmatrix} p_n & p_n' \\ q_n & q_n' \end{bmatrix}, \qquad \begin{aligned} p_n &= r^n P^n - q_n A_n, \\ p_n' &= r_n' P^n - q_n' A_n. \end{aligned}$$

Then T_n has determinant P^n and the action of T_n as a linear fractional transformation is

$$T_n z_n = \lambda \quad \text{for all } n \geqslant 0.$$

Received October 9, 1984.
1980 *Mathematics Subject Classification*. Primary 10F45.

It is clear that $|q_n\zeta + p_n|_P \leqslant P^{-n}$. Mahler showed in [5] that if $|q\zeta + p|_P \leqslant P^{-n}$, $\Phi(p, q) > 0$, then $\Phi(p, q) \geqslant \Phi(p_n, q_n)$. Thus $T(\zeta)$ determines approximations to ζ which are best with respect to $\Phi(X, Y)$. This suggests the following

Definition. The best P-adic Diophantine approximation constant is the real number c_P, the supremum over all $c > 0$ such that

$$|q\zeta + p|_P \leqslant P^{-n}, \qquad 0 < \Phi(p, q) \leqslant \frac{1}{c}P^n$$

has solutions for infinitely many n, for all $\lambda \in F$ and all P-adic integers ζ.

Since $T_n^{-1}\lambda = z_n$, it is easily seen that $y_n = \operatorname{Im} z_n = P^n/\Phi(p_n, q_n)$. Thus $z_n \in F$ implies that $y_n \geqslant \sqrt{3}/2$, and it is concluded that $c_P \geqslant \sqrt{3}/2$ for all P. Mahler showed that $c_P = \sqrt{3}/2$ if and only if $P \equiv 1 \pmod 6$. Using a result of Davenport [1], Mahler proved that $\lim_{P\to\infty} c_P = \sqrt{3}/2$. de Weger [6] generalized this last result to a larger class of norms.

For a P-adic integer ζ, set $Y(\zeta) = \overline{\lim}\, y_n$. (This value is dependent on λ.) It is clear that

$$c_P = \min_{\zeta \in \mathbf{Z}_P, \lambda \in F} Y(\zeta). \tag{1}$$

In order to study the relationship between the successive elements of $z(\zeta)$, a third sequence is introduced. For every $n \geqslant 1$, $\Omega_n \in \Omega(\zeta)$ is defined by $\Omega_n = T_{n-1}^{-1}T_n$. Each Ω_n is a matrix of determinant P with integer entries satisfying $\Omega_n z_n = z_{n-1}$. Let $M(P)$ denote the set of 2×2 integer matrices of determinant P which satisfy $\Omega F \cap F \neq \varnothing$, where Ω acts as a transformation. Any matrix which can occur in the sequence $\Omega(\zeta)$ for some P-adic integer ζ and some $\lambda \in F$ is necessarily in $M(P)$. Mahler showed that the set $M(P)$ is finite for any P and that any $\Omega \in M(P)$ can occur as some $\Omega_n \in \Omega(\zeta)$ for some P-adic integer ζ (although possibly only for special choices of λ).

Now let $m(P)$ denote the subset of $M(P)$ of integer matrices of determinant P which have a fixed point in F and trace $\equiv 0 \pmod P$. The elements of $m(P)$ are precisely those which can appear repeated successively in a sequence $\Omega(\zeta)$ for some ζ (see [5]). For $\Omega \in m(P)$, let f_Ω denote its fixed point in F. Let

$$Y(P) = \min_{\Omega \in m(P)} \operatorname{Im} f_\Omega.$$

Mahler conjectured that $c_P = Y(P)$. Moreover, he showed that for every $\varepsilon > 0$ there is a P-adic integer ζ such that $Y(\zeta) \leqslant Y(P) + \varepsilon$. So with (1) the conjecture can be stated as

MAHLER'S CONJECTURE. *For any $\lambda \in F$ and any P-adic integer ζ, $Y(\zeta) \geqslant Y(P)$.*

Mahler proved that $Y(P) = \sqrt{3}/2$ if and only if $P \equiv 1 \pmod 6$, thus for these values the conjecture is trivially verified. Mahler also verified the conjecture for $P = 2$, 3 and 5. These verifications checked that, for each $\Omega \in M(P)$, if both $z \in F$ and $\Omega z \in F$ then $\max\{\operatorname{Im} z, \operatorname{Im} \Omega z\} \geqslant Y(P)$. (For some matrices in $M(2)$ and $M(3)$ this was not the case, but such matrices could not occur more than once in succession in a sequence $\Omega(\zeta)$ for any ζ.) This suffices to verify the conjecture, for it implies that for every 5-adic integer and every index n, $\max\{y_n, y_{n+1}\} \geqslant Y(5)$. (Similarly, for any 2-adic (or 3-adic) integer and every index n, $\max\{y_n, y_{n+1}, y_{n+2}\} \geqslant Y(2)$ (or $Y(3)$).)

For this paper, the analysis is carried further by adapting the method for computer search. The conjecture was thus checked for all $P \leqslant 101$ and, in fact, verified for all such P except $P = 83$. Additional investigation showed that Mahler's conjecture is, in fact, false for $P = 83$. A periodic $\Omega(\zeta)$ sequence can be constructed as a counterexample. Additional information about periodic sequences can be found in [2], [3], [6].

This paper is organized as follows. In Section 2, the matrices in the finite set $M(P)$ are described by an explicit set of inequalities which their entries must satisfy. In Section 3, a method is described that determines a value $Y(\Omega) \leqslant \max\{\operatorname{Im} z, \operatorname{Im} \Omega z\}$ when z, $\Omega z \in F$, for each $\Omega \in M(P)$. The values of $Y(P)$ are tabulated in an appendix. If for every $\Omega \in M(P)$, $Y(\Omega) \geqslant Y(P)$, then the conjecture is verified. There are primes for which this verification fails; the first such is $P = 83$. In Section 4, this failure is discussed. In Section 5, the counterexample for $P = 83$ is given.

2. The Set $M(P)$. In this section, inequalities are given which the entries of a matrix Ω in $M(P)$ must satisfy. Since these inequalities are intended to assist in studying Mahler's conjecture, the inequalities are specifically for $P \equiv 5 \pmod 6$. The derivation of these inequalities is described and a sample derivation is given. For full details, see [2].

Throughout the paper, let

$$\Omega = \begin{bmatrix} \alpha & \alpha' \\ \beta & \beta' \end{bmatrix} \quad \text{and} \quad \rho = \frac{-1 + \sqrt{3}\, i}{2}.$$

Because Ω and $-\Omega$ represent the same transformation, the inequalities will assume that $\beta \geqslant 0$. It is clear that if $\Omega \in M(P)$ then $-P\Omega^{-1}$, an inverse of Ω as a transformation, $\in M(P)$ as well.

Definition. The *isometric circle*, I, of a transformation Ω where $\beta \neq 0$ is the circle centered at $-\beta'/\beta$ of radius $\sqrt{P}/\beta$. The isometric circle, I', of the inverse transformation is the circle centered at α/β of radius $\sqrt{P}/\beta$. (A complete discussion of isometric circles can be found in Chapter 1 of [4].)

The transformation Ω acts on the complex plane by inversion through the circle I and then through the line $x = (\alpha - \beta')/2\beta$. The circle I is mapped to the circle I'. The region inside (outside) the circle I is mapped to the region outside (inside) the circle I'.

In order to guarantee the condition $\Omega F \cap F \neq \varnothing$, it is necessary that at least one of the isometric circles I or I' intersect F. Thus ρ or $1 + \rho$ is inside one of the circles, which happens when

$$\beta^2 + \beta|\beta'| + \beta'^2 \leqslant P \quad \text{or} \quad \alpha^2 + |\alpha|\beta + \beta^2 \leqslant P. \tag{2}$$

This can be used to show that if $\beta \neq 0$, then $(\beta, \beta') = 1$.

In [5], Mahler divided the set $M(P)$ into three subsets, $M_1(P)$, $M_2(P)$ and $M_3(P)$ according to whether $\Omega F \cap F$ is a region of nonempty interior, a curve, or a single point. The elements of the sets $M_2(P)$ and $M_3(P)$ can be listed explicitly. The justification for these listings is detailed in [2].

A transformation $\Omega \in M_2(P)$ maps a portion of the boundary of F to a portion of the boundary of F. This happens for precisely the matrices

$$\begin{bmatrix} -1 & \dfrac{-(P+1)}{2} \\ 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{-(P+1)}{2} & \dfrac{(1-P)}{2} \\ 1 & -1 \end{bmatrix} \text{ and its inverse as a transformation } \begin{bmatrix} 1 & \dfrac{(1-P)}{2} \\ 1 & \dfrac{(1+P)}{2} \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{(P-1)}{2} & \dfrac{-(P+1)}{2} \\ 1 & 1 \end{bmatrix} \text{ and its inverse as a transformation } \begin{bmatrix} -1 & \dfrac{-(P+1)}{2} \\ 1 & \dfrac{(1-P)}{2} \end{bmatrix}.$$

A transformation $\Omega \in M_3(P)$ when the corner point ρ of F is mapped to a boundary point of F by Ω or its inverse. This happens precisely for the matrices

$$\begin{bmatrix} \dfrac{-(P+1)}{2} & -P \\ 1 & 0 \end{bmatrix} \text{ and its inverse as a transformation } \begin{bmatrix} 0 & -P \\ 1 & \dfrac{(1+P)}{2} \end{bmatrix}.$$

The case for $\Omega \in M_1(P)$ with $\beta = 0$ is also straightforward. These matrices are all those of the form

$$\begin{bmatrix} 1 & \alpha' \\ 0 & P \end{bmatrix} \text{ and its inverse as a transformation } \begin{bmatrix} P & -\alpha' \\ 0 & 1 \end{bmatrix}$$

for $(-P+1)/2 \leqslant \alpha' \leqslant (P-1)/2$.

Let $\mathring{F}$ denote the interior of F. $\Omega \in M_1(P)$ if and only if $\Omega\mathring{F} \cap \mathring{F} \neq \varnothing$. This condition is easier to work with because of the symmetry of $\mathring{F}$. Clearly, if Ω satisfies this condition, then so does $-P\Omega^{-1}$, the inverse transformation. Furthermore, when $\beta > 0$, the isometric circles of $\Omega' = \left[\begin{smallmatrix} \beta' & \alpha' \\ \beta & \alpha \end{smallmatrix}\right]$ and its inverse are just reflections across the imaginary axis of those of Ω^{-1} and Ω. Thus $\Omega'\mathring{F} \cap \mathring{F} \neq \varnothing$ whenever $\Omega\mathring{F} \cap \mathring{F} \neq \varnothing$. Consequently, when $\beta > 0$, of the four matrices

$$\begin{bmatrix} \alpha & \alpha' \\ \beta & \beta' \end{bmatrix}, \quad \begin{bmatrix} -\beta' & \alpha' \\ \beta & -\alpha \end{bmatrix}, \quad \begin{bmatrix} \beta' & \alpha' \\ \beta & \alpha \end{bmatrix}, \quad \begin{bmatrix} -\alpha & \alpha' \\ \beta & -\beta' \end{bmatrix}$$

it is only necessary to consider one. The inequalities for the entries of $\Omega \in M_1(P)$ with $\beta > 0$ are determined assuming that $\alpha \geqslant 0$ and $|\beta'| \geqslant \alpha$.

Assuming that $\beta > 0$ and $|\beta'| \geqslant \alpha \geqslant 0$, (2) reduces to

$$\beta^2 - \alpha\beta + \alpha^2 < P.$$

This yields an upper bound for β in terms of P,

$$\beta < \frac{2}{\sqrt{3}}\sqrt{P},$$

as well as upper and lower bounds for α in terms of β and P. However, to determine ranges for β', it is effective to break the problem into four cases:

(i) $\alpha^2 + \alpha\beta + \beta^2 < P$ and $\alpha/\beta \geqslant \frac{1}{2}$.
(ii) $\alpha^2 + \alpha\beta + \beta^2 < P$ and $\alpha/\beta < \frac{1}{2}$.
(iii) $\alpha^2 + \alpha\beta + \beta^2 > P$ and $\alpha/\beta \geqslant \frac{1}{2}$.
(iv) $\alpha^2 + \alpha\beta + \beta^2 > P$ and $\alpha/\beta < \frac{1}{2}$.

In each case, bounds for α, β and β' can be determined by examination of the image of $\mathring{F}$ under Ω^{-1}. The inequalities in case (i) are derived below as an example; for the other cases, see [2]. The inequalities for all four cases are summarized in Figure 1.

Case (i). The conditions defining case (i) imply

$$\frac{\beta}{2} \leqslant \alpha < \frac{-\beta}{2} + \frac{\sqrt{4P - 3\beta^2}}{2}.$$

This interval is nonempty when $\beta < 2\sqrt{P}/\sqrt{7}$.

If $\beta' \geqslant \alpha$, $\Omega^{-1}\mathring{F} \cap \mathring{F} \neq \varnothing$ if and only if at least one of $\Omega^{-1}(\rho)$, $\Omega^{-1}(1+\rho)$ has real part $> -\frac{1}{2}$. (Because $\alpha^2 + \alpha\beta + \beta^2 < P$, their imaginary parts are large enough.) Thus,

$$\begin{aligned} -\tfrac{1}{2} &< \max\{\operatorname{Re}\Omega^{-1}(\rho), \Omega^{-1}(1+\rho)\} \\ &= \max\left\{\frac{-\beta'}{\beta} + \frac{2\alpha+\beta}{2\beta}\frac{P}{\alpha^2+\alpha\beta+\beta^2}, \frac{-\beta'}{\beta} + \frac{2\alpha-\beta}{2\beta}\frac{P}{\alpha^2-\alpha\beta+\beta^2}\right\}, \end{aligned}$$

which becomes

$$\beta' < \max\left\{\frac{\beta}{2} + \frac{2\alpha+\beta}{2}\frac{P}{\alpha^2+\alpha\beta+\beta^2}, \frac{\beta}{2} + \frac{2\alpha-\beta}{2}\frac{P}{\alpha^2-\alpha\beta+\beta^2}\right\}.$$

The former term is larger when $\alpha < \beta$.

If $\beta' \leqslant -\alpha$, then $\Omega^{-1}\mathring{F}$ lies entirely to the right of $x = \alpha/\beta \geqslant \frac{1}{2}$ and cannot intersect $\mathring{F}$.

The entries α, β and β' of a case (i) matrix must satisfy

$$0 < \beta < \frac{2}{\sqrt{7}}\sqrt{P},$$

$$\frac{\beta}{2} \leqslant \alpha < \frac{-\beta}{2} + \frac{\sqrt{4P-3\beta^2}}{2},$$

$$\alpha \leqslant \beta' < \begin{cases} \dfrac{\beta}{2} + \dfrac{2\alpha+\beta}{2}\dfrac{P}{\alpha^2+\alpha\beta+\beta^2} & \text{if } \alpha \leqslant \beta, \\ \dfrac{\beta}{2} + \dfrac{2\alpha-\beta}{2}\dfrac{P}{\alpha^2-\alpha\beta+\beta^2} & \text{if } \alpha \geqslant \beta. \end{cases}$$

Figure 1 gives a flow diagram for deciding when a matrix Ω with $\beta > 0$ and $|\beta'| \geqslant \alpha \geqslant 0$ has $\Omega\mathring{F} \cap \mathring{F} \neq \varnothing$. Find the range on the left line where β falls. For the α scale assigned to that range, find the range in which α falls. The case inequalities named for that range give allowable choices for β'. The case inequalities are listed at the right.

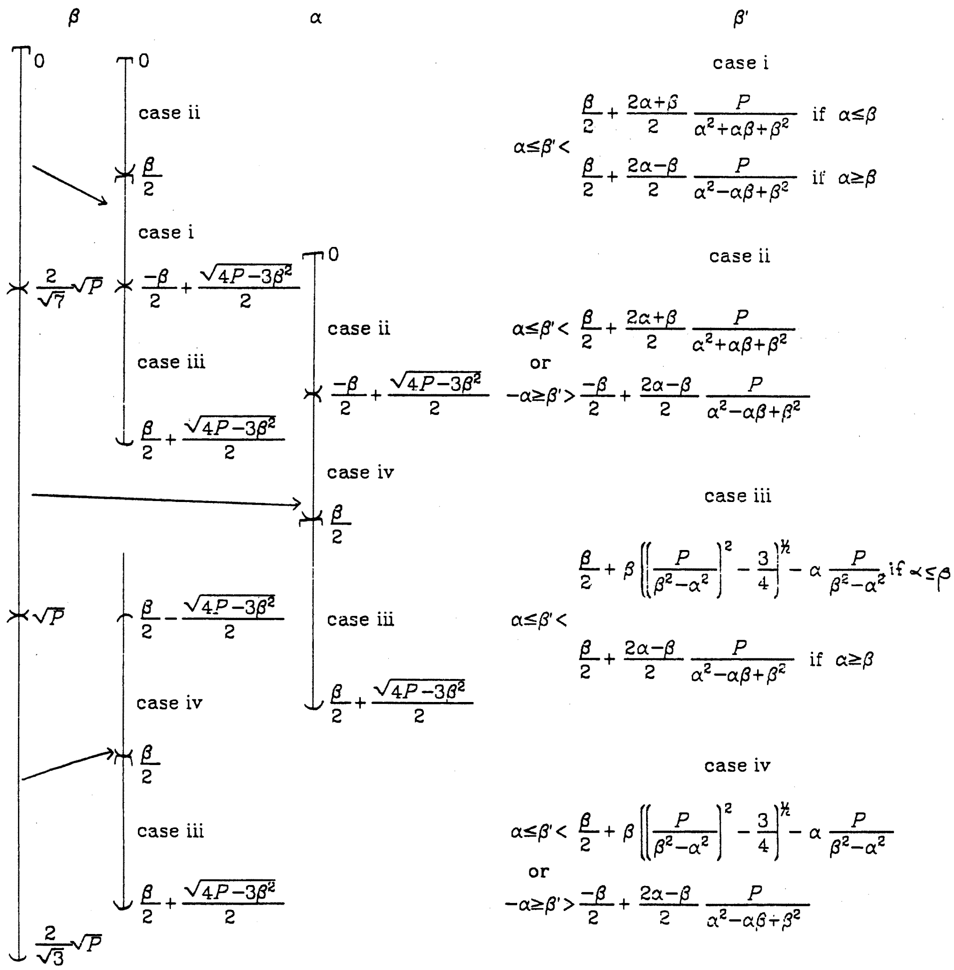

FIGURE 1

3. Testing Mahler's Conjecture. In this section, techniques used to determine the value $Y(\Omega)$ for each Ω in $M(P)$, $P \equiv 5 \pmod 6$, are discussed. The computations require an involved case analysis. Sample computations are given.

Definition. For any matrix $\Omega \in M(P)$

$$Y(\Omega) = \inf\{\max\{\operatorname{Im} z, \operatorname{Im} \Omega z\}\},$$

where the inf is taken over all z for which $z \in F$ and $\Omega z \in F$.

The values of $Y(\Omega)$ for Ω in $M_2(P)$ or $M_3(P)$ are easy to determine. These values are quite large compared to $Y(P)$ and do not figure in the conjecture testing. The values of $Y(\Omega)$ for upper-triangular Ω are also large; such matrices can thus be eliminated from serious conjecture testing work.

The complicated testing arises in computing $Y(\Omega)$ for $\Omega = \begin{bmatrix} \alpha & \alpha' \\ \beta & \beta' \end{bmatrix} \in M_1(P)$ which have $\beta > 0$. The value of $Y(\Omega)$ will be the same for the four matrices

$$\begin{bmatrix} \alpha & \alpha' \\ \beta & \beta' \end{bmatrix}, \quad \begin{bmatrix} -\alpha & \alpha' \\ \beta & -\beta' \end{bmatrix}, \quad \begin{bmatrix} \beta' & \alpha' \\ \beta & \alpha \end{bmatrix}, \quad \begin{bmatrix} -\beta' & \alpha' \\ \beta & -\alpha \end{bmatrix}$$

so only those Ω for which $\beta > 0$ and $|\beta'| \geqslant \alpha \geqslant 0$ are studied.

For $\Omega \in M_1(P)$, $Y(\Omega) = \inf\{\max\{\operatorname{Im} z, \operatorname{Im} \Omega z\}\}$, where the inf is taken over all z for which z and $\Omega z \in \bar{F}$, the closure of F. The boundary points of F frequently yield the value for $Y(\Omega)$.

The matrices are divided into cases depending on the position of the isometric circles with respect to the boundary of F.

(i) $\alpha^2 + \alpha\beta + \beta^2 < P$ and $\alpha \leqslant \beta' < \alpha + \beta$.

(ii) $\alpha^2 + \alpha\beta + \beta^2 > P$ and $\alpha \leqslant \beta' < (\alpha + \beta)/2 + (P - \beta^2)/2\alpha$.

(iii) all other matrices with $\alpha \leqslant \beta'$.

(iv) $\alpha^2 + \alpha\beta + \beta^2 < P$ and $-\alpha \geqslant \beta' > \alpha - \beta$.

(v) all other matrices with $-\alpha \geqslant \beta'$.

For each Ω, a lower bound for $Y(\Omega)$ can be established. For most matrices, the value established equals $Y(\Omega)$, but calculating this exact value is impractical in some "extreme cases", e.g., when β' is large. Two techniques are described.

For cases (i), (ii) and (iv), both the isometric circles I and I' of Ω and Ω^{-1} intersect the boundary of F. If a point z is on I (resp., I'), then $\operatorname{Im} z = \operatorname{Im} \Omega z$ (resp., $\operatorname{Im} z = \operatorname{Im} \Omega^{-1} z$). If z lies inside I (resp., I'), then $\operatorname{Im} z < \operatorname{Im} \Omega z$ (resp., $\operatorname{Im} z < \operatorname{Im} \Omega^{-1} z$). Thus a good candidate for $Y(\Omega)$ is the imaginary part of the lowest z on an isometric circle for which both z and Ωz or z and $\Omega^{-1} z$ are in $\bar{F}$. Denote this value Y_1.

It is still possible that there is a $z \in \bar{F}$ with $\operatorname{Im} z < Y_1$ and Ωz (resp., $\Omega^{-1} z) \in \bar{F}$ with $\operatorname{Im} \Omega z$ (resp., $\operatorname{Im} \Omega^{-1} z) < Y_1$. The corner points ρ and $1 + \rho$ and low boundary points of $\bar{F}$ are likely candidates for such a z. The behavior of these points must be compared with that of the isometric circle point which produced Y_1. The best comparison value is denoted Y_2. Sometimes the value easily identified for Y_2 is not the best possible, but

$$Y(\Omega) \geqslant \min\{Y_1, Y_2\}.$$

In cases (iii) and (v), the points on the isometric circles in $\bar{F}$ do not map (under Ω or Ω^{-1}) to points in $\bar{F}$, or such points are difficult to determine. A value $y(\Omega)$ may be computed instead. Let G be the set of points of the lower-boundary arcs of $\Omega^{-1}\bar{F}$ which are contained in $\bar{F}$. Set

$$y(\Omega) = \max\left\{\inf_{z \in G}\{\operatorname{Im} z\},\ \inf_{z \in G}\{\operatorname{Im} \Omega z\}\right\}.$$

Since $\inf_{z \in G}\{\operatorname{Im} z\} \leqslant Y(\Omega)$ and $\inf_{z \in G}\{\operatorname{Im} \Omega z\} \leqslant Y(\Omega)$,

$$y(\Omega) \leqslant Y(\Omega).$$

The details of case (i) are presented as an example, with figures at the end; the details of the other cases are included in [2].

Notation. The intersection point of two curves C_1 and C_2 will be denoted $C_1 \cap C_2$.

Case (i). Let

$$z_0 = I' \cap \left(x = -\tfrac{1}{2}\right),$$

$$z_0' = \begin{cases} I \cap \left(x = \tfrac{1}{2}\right) & \text{if } \beta^2 + \beta\beta' + \beta'^2 < P, \\ I \cap (|z| = 1) & \text{if } \beta^2 + \beta\beta' + \beta'^2 > P. \end{cases}$$

These are the possible isometric circle points described earlier. Set $Y_1 = \max\{\operatorname{Im} z_0, \operatorname{Im} z_0'\}$. Y_1 must be compared to the values of $\operatorname{Im} \Omega^{-1} z$, where $z, \Omega^{-1} z \in \bar{F}$ are inside I'. For each of six subcases, a value Y_2 is defined which is $\leqslant$ the least such $\operatorname{Im} \Omega^{-1} z$.

The condition

$$\beta' < \alpha + \beta < \frac{\beta}{2} + \frac{2\alpha + \beta}{2} \frac{P}{\alpha^2 + \alpha\beta + \beta^2}$$

implies that $\operatorname{Re} \Omega^{-1}(\rho) > -\frac{1}{2}$.

(a) $\Omega^{-1}(\rho) \in \bar{F}$: set $Y_2 = \operatorname{Im} \Omega^{-1}(\rho)$. Then $Y(\Omega) = \min\{Y_1, Y_2\}$.

(b) $\operatorname{Re} \Omega^{-1}(\rho) > \frac{1}{2}$: let $z_1 = \Omega^{-1}(x = -\frac{1}{2}) \cap (x = \frac{1}{2})$. Set $Y_2 = \max\{\operatorname{Im} z_1, \operatorname{Im} \Omega z_1\}$. Then $Y(\Omega) = \min\{Y_1, Y_2\}$.

In (c)–(f), $\Omega^{-1}(\rho)$ lies in the small region where $-\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}$, $\operatorname{Im} z > \sqrt{3}/2$ and $|z| < 1$. In each case, $Y_2 \leqslant Y(\Omega) \leqslant Y_1$.

(c) $\operatorname{Im}(\Omega^{-1}(x = -\frac{1}{2}) \cap (x = -\frac{1}{2})) < \sqrt{3}/2$: let $z_1 = \Omega^{-1}(x = -\frac{1}{2}) \cap (|z| = 1)$ and $z_2 = \Omega^{-1}(|z| = 1) \cap (|z| = 1)$.

In (d)–(f), $\operatorname{Im}(\Omega^{-1}(x = -\frac{1}{2}) \cap (x = -\frac{1}{2})) < \sqrt{3}/2$.

(d) $\operatorname{Re} \Omega^{-1}(1 + \rho) > \frac{1}{2}$: let $z_1 = \Omega^{-1}(|z| = 1) \cap (|z| = 1)$ and $z_2 = \Omega^{-1}(|z| = 1) \cap (x = \frac{1}{2})$.

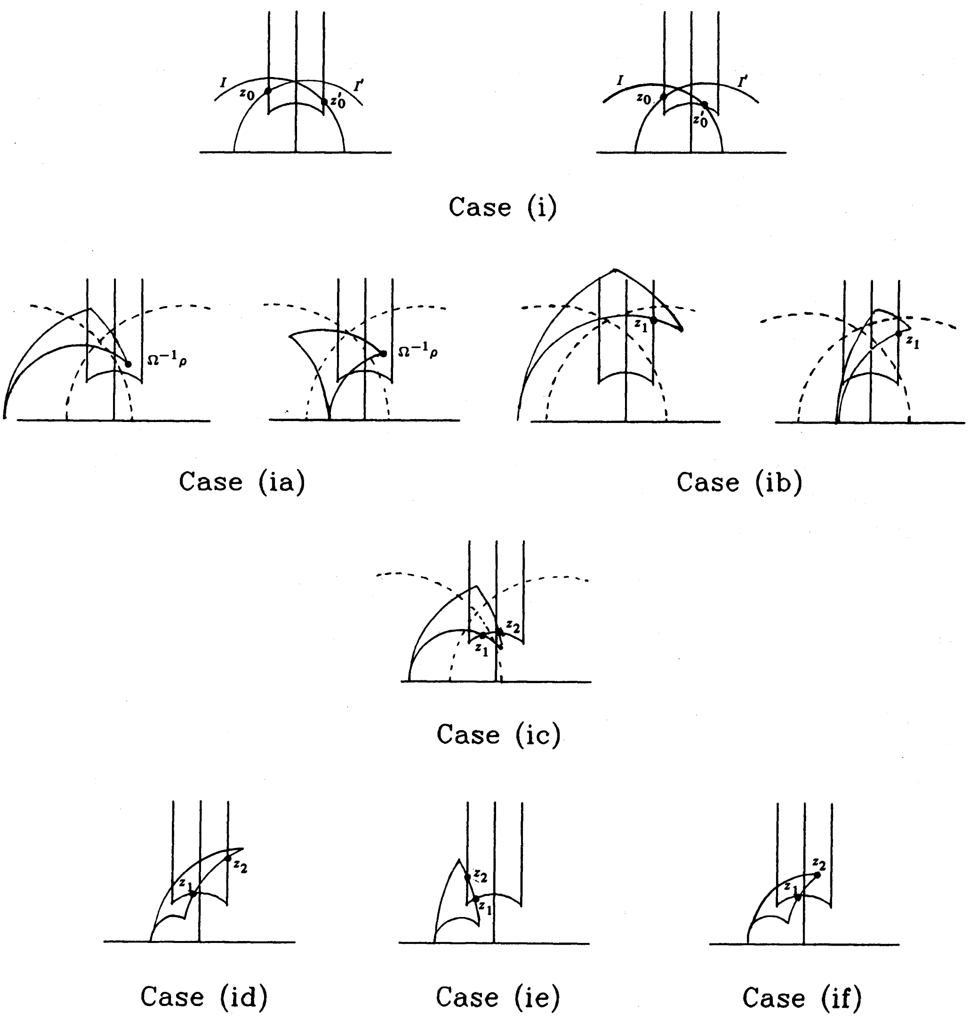

FIGURE 2

(e) $\operatorname{Re}\Omega^{-1}(1+\rho) < -\frac{1}{2}$: let $z_1 = \Omega^{-1}(|z|=1) \cap (|z|=1)$ and $z_2 = \Omega^{-1}(|z|=1) \cap (x = -\frac{1}{2})$.

(f) $\operatorname{Re}\Omega^{-1}(1+\rho) \in \bar{F}$: let $z_1 = \Omega^{-1}(|z|=1) \cap (|z|=1)$ and $z_2 = \Omega^{-1}(1+\rho)$.

In each of the above four cases, set

$$Y_2 = \max\{\min\{\operatorname{Im} z_1, \operatorname{Im} z_2\}, \min\{\operatorname{Im}\Omega z_1, \operatorname{Im}\Omega z_2\}\}.$$

4. The Results of Computer Testing. In the computational testing of Mahler's conjecture, a program was written in PASCAL, and the UNIVAC 1180 at the University of Maryland was persuaded to run it. The program tested each matrix whose entries satisfy the inequalities in Figure 1. For each such matrix Ω, a value

$$Y^*(\Omega) = \begin{cases} \min\{Y_1, Y_2\} & \text{in cases (i), (ii), and (iv)} \\ y(\Omega) & \text{in cases (iii) and (v)} \end{cases}$$

was established. In every case, $Y^*(\Omega) \leqslant Y(\Omega)$. Let $Y^*(P) = \min_\Omega Y^*(\Omega)$.

The program was run for all $P \equiv 5 \pmod 6$ up to $P = 101$. It was found that $Y^*(P) = Y(P)$ for all the values of P tested except $P = 83$, so the conjecture is true for these values.

The failure of the test for $P = 83$ does not show that the conjecture is false. The program tested the much stronger statement, that $Y(\Omega) \geqslant Y^*(\Omega) \geqslant Y(P)$ for all $\Omega \in M(P)$, and this fails for $P = 83$ in an obvious way. Consider the isometric circles I and I' for $\Omega = \left[\begin{smallmatrix} -1 & -10 \\ 9 & 7 \end{smallmatrix}\right]$ whose fixed point has imaginary part $Y(83)$. Let $z_0 = I' \cap (x = -\frac{1}{2})$; then z_0, $\Omega^{-1}z_0 \in F$ and $\operatorname{Im} z_0 = \operatorname{Im}\Omega^{-1}z_0 = \sqrt{283}/18 < \sqrt{296}/18 = Y(83)$. Such a point exists because the isometric circles intersect the boundary of F below $Y(83)$; this happens because the fixed point of Ω does not lie on the unit circle. The same problem arises for other P, whenever a fixed point f of an elliptic matrix Ω has $\operatorname{Im} f = Y(P)$ and $|f| \neq 1$. Since $|f|^2 = -\alpha'/\beta$, it is easy to identify such P by examination of the table in the appendix. For $P < 1000$, they are 83, 167, 227, 251, 359, 467, 479, 587, 647, 743, 773, 797, 827, 911, 941 and 947.

5. A Counterexample. It is not possible to disprove the conjecture using only the fact that $Y(\Omega) < Y(83)$ for the matrix $\left[\begin{smallmatrix} -1 & -10 \\ 9 & 7 \end{smallmatrix}\right]$. However, the computer testing found four other matrices satisfying the inequalities in Figure 1 for which $Y(\Omega) < Y(83)$; they are

$$\begin{bmatrix} 3 & -8 \\ 7 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & -7 \\ 8 & 9 \end{bmatrix}, \quad \begin{bmatrix} 7 & -9 \\ 3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 7 & -3 \\ 9 & 8 \end{bmatrix}.$$

This yields 20 matrices in $M_1(83)$ having $Y(\Omega) < Y(83)$. An examination of products of pairs of these matrices found

$$\begin{bmatrix} -9 & -8 \\ 7 & -3 \end{bmatrix}\begin{bmatrix} -9 & -7 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix},$$

an elliptic matrix whose fixed point $-19 + \sqrt{3003}\,i/58$ has imaginary part less than $Y(83)$, and

$$\begin{bmatrix} -9 & -7 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} -9 & -8 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 32 & 93 \\ -93 & -55 \end{bmatrix},$$

an elliptic matrix whose fixed point $(-29 + \sqrt{3003}\,i)/62$ has imaginary part less than $Y(83)$. For a specific $\lambda \in F$, a periodic sequence of the above matrix factors can be shown to be the $\Omega(\zeta)$ sequence of a P-adic integer ζ with $Y(\zeta) < Y(83)$, disproving the conjecture.

In [5], Mahler proved that a sequence $\{\Omega_n\}_{n=1}^{\infty}$ is the sequence $\Omega(\zeta)$ for some ζ if and only if, for a given λ, and setting

$$T_n = \Omega_1 \cdots \Omega_n = \begin{bmatrix} p_n & p'_n \\ q_n & q'_n \end{bmatrix},$$

$T_n^{-1}\lambda \in F$ and $(q_n, q'_n) = 1$.

For the counterexample, let

$$\lambda = \frac{-19 + \sqrt{3003}\, i}{58}$$

and let

$$\Omega_n = \begin{cases} \begin{bmatrix} -9 & -8 \\ 7 & -3 \end{bmatrix} & \text{for } n \text{ odd}, \\ \begin{bmatrix} -9 & -7 \\ 8 & -3 \end{bmatrix} & \text{for } n \text{ even}. \end{cases}$$

Then, by definition,

$$T_n = \begin{cases} \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix}^{(n-1)/2} \begin{bmatrix} -9 & -8 \\ 7 & -3 \end{bmatrix} & \text{for } n \text{ odd}, \\ \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix}^{n/2} & \text{for } n \text{ even}. \end{cases}$$

Note that

$$T_n^{-1}\lambda = \begin{cases} \dfrac{-29 + \sqrt{3003}\, i}{62} \in F & \text{for } n \text{ odd}, \\ \dfrac{-19 + \sqrt{3003}\, i}{58} = \lambda \in F & \text{for } n \text{ even}. \end{cases}$$

Furthermore, by a congruence relation in Section 9 of [5], for even n,

$$T_n = T_{2k} = \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix}^k \equiv (17 - 40)^{k-1} \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix} \pmod{83},$$

hence q_n and q'_n are not both divisible by 83 and must be relatively prime. Also, for odd n,

$$\begin{aligned} T_n = T_{2k+1} &= \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix}^k \begin{bmatrix} -9 & -8 \\ 7 & -3 \end{bmatrix} \\ &\equiv (17 - 40)^{k-1} \begin{bmatrix} 17 & 87 \\ -87 & -40 \end{bmatrix} \begin{bmatrix} -9 & -8 \\ 7 & -3 \end{bmatrix} \pmod{83} \end{aligned}$$

so that again, $(q_n, q'_n) = 1$. This verifies that the sequence $\{\Omega_n\}$ is a legitimate $\Omega(\zeta)$ sequence; it is in fact the sequence for $\zeta = -\lambda = (19 - \sqrt{3003}\, i)/58$ with the indicated λ. This is a counterexample to the conjecture because

$$Y(\zeta) = \overline{\lim}\ \mathrm{Im}\, T_n^{-1}\lambda = \max\left\{ \frac{\sqrt{3003}}{58}, \frac{\sqrt{3003}}{62} \right\} < \frac{\sqrt{74}}{9} = Y(83).$$

It is likely that such constructions exist for other P for which $Y(P)$ occurs as the imaginary part of a fixed point of a matrix in $m(P)$ where the fixed point is not on the unit circle.

The nature of the construction of the counterexample suggests that the value of the best P-adic Diophantine approximation constant may be determined by the imaginary part of a fixed point in F on the unit circle of an elliptic matrix of determinant P^g. This possibility is being studied.

Acknowledgments. The author would like to thank W. Adams for suggesting the problem and for his help and encouragement before, during and after the preparation of the dissertation in which much of this work orginally appeared. The author is also indebted to D. Shanks for his probing comments on the validity of Mahler's conjecture.

Appendix. The table below gives $Y(P)$ for all $P \equiv 5 \pmod 6$ less than 1000, along with a matrix in $m(P)$ whose fixed point has imaginary part $Y(P)$.

P	$Y(P)$	matrix	P	$Y(P)$	matrix	P	$Y(P)$	matrix
5	1	$\begin{bmatrix}2&-1\\1&2\end{bmatrix}$	173	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}4&-11\\11&13\end{bmatrix}$	389	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix}10&-13\\13&22\end{bmatrix}$
11	$\frac{\sqrt{35}}{6}$	$\begin{bmatrix}1&-3\\3&2\end{bmatrix}$	179	$\frac{\sqrt{275}}{18}$	$\begin{bmatrix}7&-9\\9&14\end{bmatrix}$	401	$\frac{\sqrt{280}}{19}$	$\begin{bmatrix}2&-19\\19&20\end{bmatrix}$
17	$2\frac{\sqrt{2}}{3}$	$\begin{bmatrix}2&-3\\3&4\end{bmatrix}$	191	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}5&-11\\11&14\end{bmatrix}$	419	$\frac{\sqrt{1675}}{46}$	$\begin{bmatrix}-11&-23\\23&10\end{bmatrix}$
23	$\frac{\sqrt{91}}{10}$	$\begin{bmatrix}-2&-5\\5&1\end{bmatrix}$	197	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix}2&-13\\13&14\end{bmatrix}$	431	$\frac{\sqrt{1675}}{46}$	$\begin{bmatrix}-14&-23\\23&7\end{bmatrix}$
29	$\frac{\sqrt{91}}{10}$	$\begin{bmatrix}1&-5\\5&4\end{bmatrix}$	227	$\frac{\sqrt{191}}{15}$	$\begin{bmatrix}-13&-16\\15&1\end{bmatrix}$	443	$\frac{\sqrt{931}}{34}$	$\begin{bmatrix}7&-17\\17&22\end{bmatrix}$
41	$2\frac{\sqrt{10}}{7}$	$\begin{bmatrix}-4&-7\\7&2\end{bmatrix}$	233	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix}4&-13\\13&16\end{bmatrix}$	449	$\frac{\sqrt{280}}{19}$	$\begin{bmatrix}4&-19\\19&22\end{bmatrix}$
47	$\frac{\sqrt{187}}{14}$	$\begin{bmatrix}-2&-7\\7&1\end{bmatrix}$	239	$\frac{\sqrt{931}}{34}$	$\begin{bmatrix}-10&-17\\17&5\end{bmatrix}$	461	$\frac{\sqrt{1675}}{46}$	$\begin{bmatrix}-17&-23\\23&4\end{bmatrix}$
53	$\frac{\sqrt{91}}{10}$	$\begin{bmatrix}4&-5\\5&7\end{bmatrix}$	251	$\frac{\sqrt{242}}{17}$	$\begin{bmatrix}-11&-18\\17&5\end{bmatrix}$	467	$\frac{\sqrt{431}}{23}$	$\begin{bmatrix}-17&-24\\23&5\end{bmatrix}$
59	$\frac{\sqrt{55}}{8}$	$\begin{bmatrix}-5&-8\\8&1\end{bmatrix}$	257	$\frac{\sqrt{176}}{15}$	$\begin{bmatrix}2&-15\\15&16\end{bmatrix}$	479	$\frac{\sqrt{479}}{24}$	$\begin{bmatrix}-11&-25\\24&11\end{bmatrix}$
71	$\frac{\sqrt{275}}{18}$	$\begin{bmatrix}-5&-9\\9&2\end{bmatrix}$	263	$\frac{\sqrt{931}}{34}$	$\begin{bmatrix}-13&-17\\17&2\end{bmatrix}$	491	$\frac{\sqrt{455}}{24}$	$\begin{bmatrix}-17&-24\\24&5\end{bmatrix}$
83	$\frac{\sqrt{74}}{9}$	$\begin{bmatrix}-7&-10\\9&1\end{bmatrix}$	269	$\frac{\sqrt{65}}{9}$	$\begin{bmatrix}-11&-18\\18&5\end{bmatrix}$	503	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}1&-22\\22&19\end{bmatrix}$
89	$2\frac{\sqrt{10}}{7}$	$\begin{bmatrix}4&-7\\7&10\end{bmatrix}$	281	$\frac{\sqrt{280}}{19}$	$\begin{bmatrix}-10&-19\\19&8\end{bmatrix}$	509	$\frac{\sqrt{1675}}{23}$	$\begin{bmatrix}-20&-23\\23&1\end{bmatrix}$
101	$\frac{\sqrt{65}}{9}$	$\begin{bmatrix}2&-9\\9&10\end{bmatrix}$	293	$\frac{\sqrt{731}}{30}$	$\begin{bmatrix}4&-15\\15&17\end{bmatrix}$	521	$\frac{\sqrt{40}}{7}$	$\begin{bmatrix}13&-14\\14&25\end{bmatrix}$
107	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}-7&-11\\11&2\end{bmatrix}$	311	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}10&-11\\11&19\end{bmatrix}$	557	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix}-17&-26\\26&7\end{bmatrix}$
113	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}-8&-11\\11&1\end{bmatrix}$	317	$\frac{\sqrt{1219}}{38}$	$\begin{bmatrix}-11&-19\\19&4\end{bmatrix}$	563	$\frac{\sqrt{731}}{30}$	$\begin{bmatrix}13&-15\\15&26\end{bmatrix}$
131	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix}1&-11\\11&10\end{bmatrix}$	347	$\frac{\sqrt{1219}}{38}$	$\begin{bmatrix}-14&-19\\19&1\end{bmatrix}$	569	$\frac{\sqrt{560}}{27}$	$\begin{bmatrix}-16&-27\\27&10\end{bmatrix}$
137	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix}-8&-13\\13&4\end{bmatrix}$	353	$\frac{\sqrt{1403}}{42}$	$\begin{bmatrix}-11&-21\\21&8\end{bmatrix}$	587	$\frac{\sqrt{587}}{27}$	$\begin{bmatrix}-13&-28\\27&13\end{bmatrix}$
149	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix}-10&-13\\13&2\end{bmatrix}$	359	$\frac{\sqrt{278}}{18}$	$\begin{bmatrix}1&-19\\18&17\end{bmatrix}$	593	$\frac{\sqrt{2291}}{54}$	$\begin{bmatrix}-17&-27\\27&8\end{bmatrix}$
167	$\frac{\sqrt{131}}{12}$	$\begin{bmatrix}1&-13\\12&2\end{bmatrix}$	383	$\frac{\sqrt{319}}{20}$	$\begin{bmatrix}-17&-20\\20&1\end{bmatrix}$	599	$\frac{\sqrt{455}}{24}$	$\begin{bmatrix}1&-24\\24&23\end{bmatrix}$

P	$Y(P)$	matrix	P	$Y(P)$	matrix	P	$Y(P)$	matrix
617	$\frac{\sqrt{176}}{15}$	$\begin{bmatrix} 14 & -15 \\ 15 & 28 \end{bmatrix}$	761	$\frac{\sqrt{736}}{31}$	$\begin{bmatrix} -20 & -31 \\ 31 & 10 \end{bmatrix}$	887	$\frac{\sqrt{403}}{22}$	$\begin{bmatrix} 13 & -22 \\ 22 & 31 \end{bmatrix}$
641	$\frac{\sqrt{560}}{27}$	$\begin{bmatrix} -22 & -27 \\ 27 & 4 \end{bmatrix}$	773	$\frac{\sqrt{191}}{15}$	$\begin{bmatrix} -17 & -32 \\ 30 & 11 \end{bmatrix}$	911	$\frac{\sqrt{767}}{31}$	$\begin{bmatrix} -27 & -32 \\ 31 & 3 \end{bmatrix}$
647	$\frac{\sqrt{643}}{28}$	$\begin{bmatrix} -15 & -29 \\ 28 & 11 \end{bmatrix}$	797	$\frac{\sqrt{796}}{31}$	$\begin{bmatrix} -15 & -32 \\ 31 & 13 \end{bmatrix}$	929	$\frac{\sqrt{176}}{15}$	$\begin{bmatrix} 1 & -30 \\ 30 & 29 \end{bmatrix}$
653	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix} -23 & -26 \\ 26 & 1 \end{bmatrix}$	809	$\frac{\sqrt{280}}{19}$	$\begin{bmatrix} 14 & -19 \\ 19 & 32 \end{bmatrix}$	941	$\frac{\sqrt{215}}{16}$	$\begin{bmatrix} -23 & -33 \\ 32 & 5 \end{bmatrix}$
659	$\frac{\sqrt{2635}}{58}$	$\begin{bmatrix} -14 & -29 \\ 29 & 13 \end{bmatrix}$	821	$\frac{\sqrt{133}}{13}$	$\begin{bmatrix} 5 & -26 \\ 26 & 29 \end{bmatrix}$	947	$\frac{\sqrt{866}}{33}$	$\begin{bmatrix} -25 & -34 \\ 33 & 7 \end{bmatrix}$
677	$\frac{\sqrt{481}}{25}$	$\begin{bmatrix} 2 & -25 \\ 25 & 26 \end{bmatrix}$	827	$\frac{\sqrt{2867}}{58}$	$\begin{bmatrix} -24 & -31 \\ 29 & 3 \end{bmatrix}$	953	$\frac{\sqrt{3811}}{70}$	$\begin{bmatrix} -17 & -35 \\ 35 & 16 \end{bmatrix}$
683	$\frac{\sqrt{2291}}{54}$	$\begin{bmatrix} -23 & -27 \\ 27 & 2 \end{bmatrix}$	839	$\frac{\sqrt{1675}}{46}$	$\begin{bmatrix} 10 & -23 \\ 23 & 31 \end{bmatrix}$	971	$\frac{\sqrt{1675}}{46}$	$\begin{bmatrix} 13 & -23 \\ 23 & 34 \end{bmatrix}$
701	$\frac{\sqrt{2635}}{58}$	$\begin{bmatrix} -20 & -29 \\ 29 & 7 \end{bmatrix}$	857	$\frac{\sqrt{736}}{31}$	$\begin{bmatrix} -26 & -31 \\ 31 & 4 \end{bmatrix}$	977	$\frac{\sqrt{833}}{33}$	$\begin{bmatrix} -28 & -33 \\ 33 & 4 \end{bmatrix}$
719	$\frac{\sqrt{319}}{20}$	$\begin{bmatrix} 11 & -20 \\ 20 & 29 \end{bmatrix}$	863	$\frac{\sqrt{799}}{32}$	$\begin{bmatrix} -23 & -32 \\ 32 & 7 \end{bmatrix}$	983	$\frac{\sqrt{3811}}{70}$	$\begin{bmatrix} -22 & -35 \\ 35 & 11 \end{bmatrix}$
743	$\frac{\sqrt{734}}{30}$	$\begin{bmatrix} -17 & -31 \\ 30 & 11 \end{bmatrix}$	881	$\frac{\sqrt{481}}{25}$	$\begin{bmatrix} 8 & -25 \\ 25 & 32 \end{bmatrix}$			

Department of Mathematical Sciences
Villanova University
Villanova, Pennsylvania 19085

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